Self-organization in cellular automata: a particle-based approach

Benjamin Hellouin de Menibus[1] and Mathieu Sablik[2]

[1] Laboratoire d'Informatique Fondamentale, Université de Provence, Marseille, France, benjamin.hellouin-de-menibus@ens-lyon.fr
[2] Laboratoire d'Informatique Fondamentale, Université de Provence, Marseille, France, sablik@latp.univ-mrs.fr

Abstract. For some classes of cellular automata, we observe empirically a phenomenon of self-organization: starting from a random configuration, regular strips separated by defects appear in the space-time diagram. When there is no creation of defects, all defects have the same direction after some time. In this article, we propose to formalise this phenomenon. Starting from the notion of propagation of defect by a cellular automaton formalized in [Piv07b,Piv07a], we show that, when iterating the automaton on a random configuration, defects in one direction only remain asymptotically.

1 Introduction

Cellular automata (CA) were introduced by J. von Neumann and S. Ulam as simplified models of biological systems which can exhibit self-reproduction and universal computation. A cellular automaton is a complex system defined by a local rule which acts synchronously and uniformly on the configuration space. These simple models have a wide variety of dynamical behaviours.

A first empirical classification was suggested by S. Wolfram [Wol84]. He introduced four behaviour classes and we are interested in the fourth one: *"The fourth class of cellular automata exhibits still more complicated behaviour [...]. Even starting from disordered or random initial configurations, cellular automata evolve to generate characteristic patterns. Such self-organizing behaviour occurs by virtue of the irreversibility of cellular automaton evolution."* Indeed, for some cellular automata, starting from a random configuration we observe the emergence and the persistence of homogeneous regions separated by particles which propagate and sometimes collide over time (see Fig. 1).

The persistence of these regions under the action of a CA was studied empirically [Wol84,BNR91,HC97] and theoretically [Elo94]. M. Pivato proposed a general formalism to describe this phenomenon: regions are characterized by a subshift Σ and particles are defects in a configuration of Σ. In particular, he develops some invariants to characterize the persistence of a defect [Piv07b] and he describes the different dynamics of propagation of a defect [Piv07a].

To explain the emergence of a particular subshift when starting from a configuration chosen randomly according to a measure μ, P. Kůrka and A. Maass introduced the μ-limit set, which is the subshift whose forbidden patterns are exactly those for which the probability to appear tends to zero as time tends to infinity [KM00]. This set corresponds to the configurations observed when a random configuration is iterated. The μ-limit set of an arbitrary CA is difficult to compute: for example, it is undecidable to determine if it contains only one configuration [BPT06]. In order to compute it in some given cases, P. Kůrka suggests an approach based on particle weight function which assigns weights to certain words [K̊03]. However, this method does not explain why some defects remain in the μ-limit set.

In this article we combine the notions of defect of a subshift Σ and μ-limit set to explain how structures can emerge from interactions of defects. More precisely, we show that for a subshift Σ such that defects have good collision properties, only defects in one particular direction can remain in the μ-limit set for a given σ-ergodic measure μ. In the last section, we show that this theorem can be applied to different cellular automata, thus explaining the behaviours observed in the examples shown in Fig. 1.

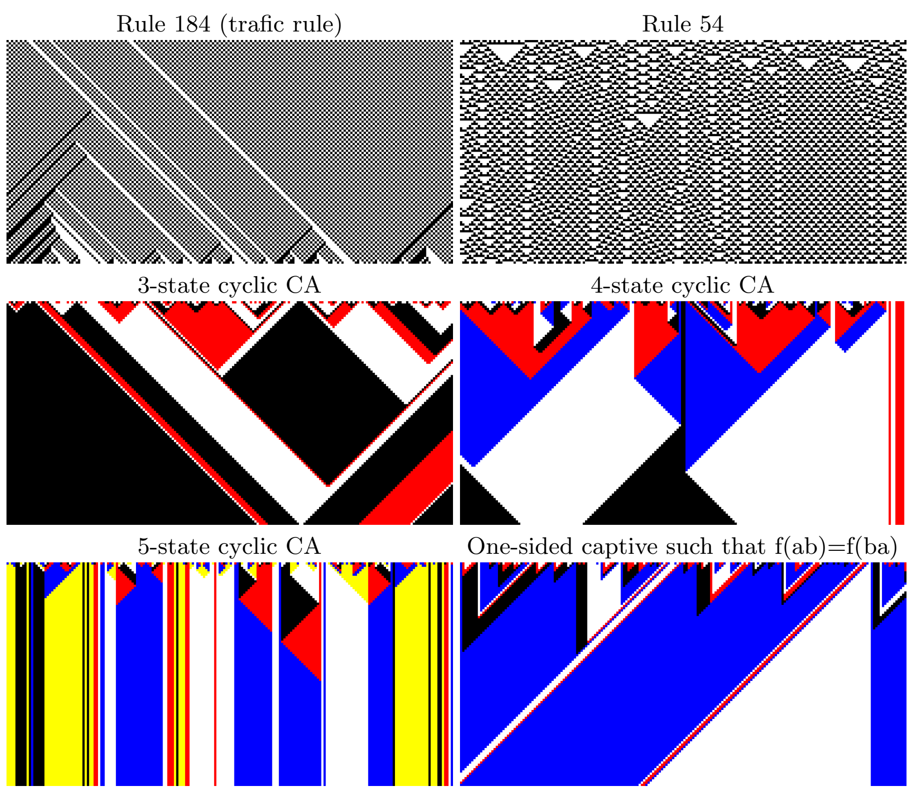

Fig. 1. Space-time diagrams of some cellular automata starting from a random configuration

2 Definitions

2.1 Configurations and cellular automata

Let $\mathcal{A}$ be a finite alphabet. We consider the spaces $\mathcal{A}^* = \cup_{n\in\mathbb{N}}\mathcal{A}^{[0,n]}$ of finite *words* and $\mathcal{A}^{\mathbb{Z}}$ of bi-infinite *configurations*. For $n \in \mathbb{N}, u \in \mathcal{A}^*, a \in \mathcal{A}^{\mathbb{Z}}$, we write $u \sqsubset_n a$ for $a_{[n,n+|u|-1]} = u$ and $u \sqsubset a$ for $\exists n, u \sqsubset_n a$. The product topology on $\mathcal{A}^{\mathbb{Z}}$ is metrizable with the *Cantor metric* on $\mathcal{A}^{\mathbb{Z}}$ defined by $d(a,b) = 2^{-\Delta(a,b)}$, where $\Delta(a,b) = \min\{|z| : z \in \mathbb{Z} \text{ and } a_z \neq b_z\}$. For $u \in \mathcal{A}^*$ and $m \in \mathbb{Z}$, we define the *cylinder* $[u]_m = \{a \in \mathcal{A}^{\mathbb{Z}} : u \sqsubset_m a\}$, if $m = 0$ we denote $[u] = [u]_0$. Cylinders are clopen sets and a base for the Cantor topology. If $U \subseteq \mathcal{A}^*$, we also note $[U]_m = \bigcup_{u\in U}[u]_m \subseteq \mathcal{A}^{\mathbb{Z}}$ which is a borelian and $[U] = [U]_0$.

The *shift* function $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $(\sigma(a))_v = a_{v+1}$ for all $v \in \mathbb{Z}$. The *language* of a set $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ is $\mathcal{L}(\Sigma) = \{u \in \mathcal{A}^* : \exists x \in \Sigma, u \sqsubset x\}$. Also, we note $\mathcal{L}_r(\Sigma) = \mathcal{L}(\Sigma) \cap \mathcal{A}^{[0,r-1]}$. A subset $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}}$ is a *subshift* if it is closed for Cantor topology and σ-invariant (i.e. $\sigma(\Sigma) \subseteq \Sigma$). In particular, Σ is a *subshift of finite type (SFT)* if there is an order $r > 0$ such that Σ is entirely defined by $\mathcal{L}_r(\Sigma)$, in the sense that $\Sigma = \{a \in \mathcal{A}^{\mathbb{Z}} : \forall z \in \mathbb{Z}, a_{[z,z+r-1]} \in \mathcal{L}_r(\Sigma)\}$. A subshift is *transitive* if there exists an $a \in \Sigma$ such that the orbit $\{\sigma^z(a)\}_{z\in\mathbb{N}}$ is dense in Σ. For a word $u \in \mathcal{A}^*$, we note ${}^{\infty}u^{\infty}$ the σ-periodic configuration of period $|u|$ such that $({}^{\infty}u^{\infty})_{[0,|u|-1]} = u$.

A *cellular automaton* is a continuous function $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ which commutes with σ. Equivalently [Hed69], F is defined by a *local rule* $f : \mathcal{A}^{\mathbb{B}_r} \to \mathcal{A}$ such that $F(a)_z = f(a_{z+\mathbb{B}_r})$ (r is the *radius* of the automaton) where $\mathbb{B}_r = [-r, r]$. We study the action of F on $\mathcal{A}^{\mathbb{Z}}$, and especially the values of $(F^n(x))_{n\in\mathbb{N}}$ for some initial configuration x, which we represent as a *space-time diagram*.

2.2 Measures and density of configuration

We consider probability measures on the borelians of $\mathcal{A}^{\mathbb{Z}}$, noted $\mathcal{M}(\mathcal{A}^{\mathbb{Z}})$. For some property P, if $\mu(\{x \in \mathcal{A}^{\mathbb{Z}} : P(x)\}) = 1$, we say that P is true for μ-almost all x.

A probability measure μ is *σ-invariant* if for any borelian set X, we have $\mu(\sigma(X)) = \mu(X)$. A probability measure μ is *σ-ergodic* if it is σ-invariant and if for any σ-invariant borelian $X \subseteq \mathcal{A}^{\mathbb{Z}}$ (i.e. $\sigma(X) \subseteq X$) one has $\mu(X) = 0$ or 1. We call $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ and $\mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$, respectively, the set of σ-invariant measures and the set of σ-ergodic measures. Of course $\mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}}) \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \subset \mathcal{M}(\mathcal{A}^{\mathbb{Z}})$.

For a configuration $a \in \mathcal{A}^{\mathbb{Z}}$, we define the *Dirac measure* as $\delta_a(U) = 1$ if $a \in U$ and 0 if not for any borelian U. We also define the *Bernoulli measure* μ associated at a sequence $(p_a)_{a\in\mathcal{A}}$ (which verifies $\sum_{a\in\mathcal{A}} p_a = 1$) as $\mu([u]_0) = p_{u_0}p_{u_1}\cdots p_{u_{|u|-1}}$. The action of F on a probability measure μ is $F\mu(X) = \mu(F^{-1}(X))$ for any borelian X. Thus we obtain a function $F : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \to \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, with $F(\mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})) \subseteq \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$.

Define the *density* of $\mathbb{U} \subseteq \mathbb{Z}$ as $d_{\mathbb{U}} = \limsup \frac{1}{2n+1}|\mathbb{U} \cap \mathbb{B}_n|$. For a set $U \subseteq \mathcal{A}^*$ and $a \in \mathcal{A}^{\mathbb{Z}}$ denote $\mathbb{U}(a) = \{n \in \mathbb{Z} : \exists u \in U, u \sqsubset_n a\} \subseteq \mathbb{Z}$ the *set of positions of U in a*. For $\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$ and if U is a finite set of words, the Birkhoff ergodic theorem [Wal00] applied to characteristic functions of cylinders can be restated in terms of density:

$$\text{For } \mu\text{-almost all } a \in \mathcal{A}^{\mathbb{Z}}, d_U(a) = d_{\mathbb{U}(a)} = \limsup_{n\to\infty} \frac{1}{2n+1} |\mathbb{U}(a) \cap \mathbb{B}_n| = \mu([U]).$$

Moreover, we also have for any two open sets A and B:

$$\frac{1}{N}\sum_{k=0}^{N} \mu(A \cap \sigma^{-k}(B)) \underset{N\to\infty}{\longrightarrow} \mu(A)\cdot\mu(B)$$

2.3 Limit and μ-limit sets

The study of self-organization leads to an interest in the behaviour of the cellular automaton when time tends to infinity. The set of configurations which appear infinitely often is the *limit set* of F defined by $\Omega(F) = \bigcap_{n=0}^{\infty} F^n(\mathcal{A}^{\mathbb{Z}})$. This set can be viewed as the largest attractor: a closed set A is an *attractor* if there exists an open set X such that $F(\overline{X}) \subset X$ and $A = \bigcap_{n=0}^{\infty} F^n(X)$ [Hur90a].

However, these topological notions do not capture the empirical point of view where the initial configuration is randomly chosen according to a measure μ. That is why the notion of μ-attractor is introduced by [Hur90b]: for $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$, a closed set A is a *μ-attractor* if A is an attractor of X and $\mu(X) > 0$. As discussed in [KM00] with many examples, this notion is not satisfactory empirically and the authors introduced the notion of *μ-limit set*:

$$\Lambda_\mu(F) = \left\{ x \in \mathcal{A}^{\mathbb{Z}} : \forall u \sqsubset x, F^n\mu([u]_0) \underset{n\to+\infty}{\not\longrightarrow} 0 \right\}.$$

3 Defects

In this section, we recall the formalism introduced in [Piv07b,Piv07a] to describe defects with respect to a subshift Σ, and we introduce a formalism to study their dynamics under the action of a cellular automaton. More precisely, we focus our study on interfaces and dislocations.

3.1 General definitions

The *defect field* of $a \in \mathcal{A}^{\mathbb{Z}}$ with respect to a subshift Σ is defined for all $z \in \mathbb{Z}$ by $\mathcal{F}_a^{\Sigma}(z) = \max\left\{ r \in \mathbb{N} : a_{z+[-\lfloor \frac{r-1}{2} \rfloor, \lfloor \frac{r}{2} \rfloor]} \in \mathcal{L}_r(\Sigma) \right\}$ where the result is possibly 0 or ∞ if the set is empty or infinite. Intuitively, this function returns the size of the largest admissible word centered on a cell. The set of *defects* $\mathbb{D}^{\Sigma}(a)$ is the set of local minima of $\mathcal{F}_a^{\Sigma}$. The *successor* of $d \in \mathbb{D}^{\Sigma}(a)$ is $s_{\mathbb{D}^{\Sigma}(a)}(d) = \min\{z \in \mathbb{D}^{\Sigma}(a) : z > d\}$, and the interval $[d+1, s_{\mathbb{D}^{\Sigma}(a)}(d)]$ is a *homogeneous region* in the sense that $a_{[d+1, s_{\mathbb{D}^{\Sigma}(a)}(d)]} \in \mathcal{L}(\Sigma)$. If there is no ambiguity, we just write $\mathbb{D}$ and $s(d)$.

If Σ is a SFT of order r, any defect d of a satisfies $\mathcal{F}_a(d) \leq r$, thus this notion can be extended to finite words of size $\geq r$ except for the first $\lfloor \frac{r-1}{2} \rfloor$ and last $\lfloor \frac{r}{2} \rfloor$ cells. Thus for a word $u \in \mathcal{A}^{[0,n-1]}$, we have $\mathbb{D}(u) \subseteq [\lfloor \frac{r-1}{2} \rfloor; n-1-\lfloor \frac{r}{2} \rfloor]$.

The examples given in Fig. 1 suggest that, in each case, defects can be classified according to their behaviour in two ways:

- Regions correspond to different subshifts and defects behave according to their surrounding regions (*interfaces* - e.g. cyclic automaton);
- Regions correspond to the same periodic subshift and defects correspond to a "phase transition" (*dislocations* - e.g. rule 184 automaton).

3.2 Interfaces

We now assume that the subshift Σ can be decomposed as a disjoint union $\Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ of σ-transitive SFT. Since the different *domains* $(\Sigma_k)_{k\in[1,n]}$ are disjoint SFTs, Σ is also an SFT, and there is some $\alpha > 0$ such that $u \in \mathcal{L}_\alpha(\Sigma) \Leftrightarrow \exists! k, u \in \mathcal{L}(\Sigma_k)$: we say that *$u$ belongs to the domain k*. Thus, for a configuration a, we can associate a domain with each homogeneous region $[d+1; s(d)]$, and only one choice is possible if $s(d) > d + \alpha$.

A *domain signature* is a continuous σ-invariant function $\kappa_d : \mathcal{A}^{\mathbb{Z}} \to \{1 \dots n\}^{\mathbb{Z}}$ that satisfies the following conditions:

- $\kappa_d(a)_z \neq \kappa_d(a)_{z+1}$ only if $z \in \mathbb{D}(a)$;
- if $\forall z \in [d+1, s(d)], \kappa_d(a)_z = k$, then $a_{[d+1,s(d)]} \in \mathcal{L}(\Sigma_k)$.

We can classify interfaces according to the domain signatures of the surrounding regions: we write $\mathbb{D}^{\kappa_d}_{i,j}(a) = \{d \in \mathbb{D}(a) : \kappa_d(a)_d = i, \kappa_d(a)_{d+1} = j\}$. It is possible to define those sets for finite words except for the first $\alpha - 1$ and last α cells.

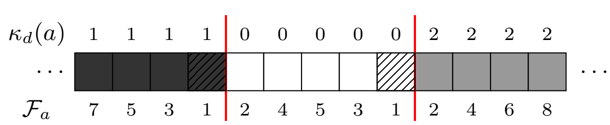

Fig. 2. Interfaces between monochromatic domains

3.3 Dislocations

Let Σ be a σ-transitive SFT of order $r > 1$. We say that Σ is *P-periodic* if there exists a partition $V_1, \dots, V_P$ of $\mathcal{L}_{r-1}(\Sigma)$ such that $a_1, \dots, a_r \in \mathcal{L}_r(\Sigma)$ if and only if there exists $i \in \mathbb{Z}/P\mathbb{Z}$ such that $a_1, \dots, a_{r-1} \in V_i$ and $a_2, \dots, a_r \in V_{i+1}$. The *period* of Σ is the maximal $P \in \mathbb{N}$ such that Σ is P-periodic.

We thus associate to each $a \in \Sigma$ its *phase* $\varphi(a) \in \mathbb{Z}/P\mathbb{Z}$ such that $a_{[0,r-2]} \in V_{\varphi(a)}$. Obviously, $\varphi(\sigma^k(a)) = \varphi(a) + k$. For $a \in \mathcal{A}^{\mathbb{Z}}$, we say that the homogeneous region $[d+1, s(d)]$ *is in phase k* if $\exists b \in \Sigma, \varphi(b) = k, a \sqsubset_{d+1} b$. If $s(d) > d + r - 2$, the phase is unique and corresponds to $a_{[d+1,d+r-1]} \in V_{k+d+1}$.

A *phase signature* $\kappa_\varphi : \mathcal{A}^{\mathbb{Z}} \to (\mathbb{Z}/P\mathbb{Z})^{\mathbb{Z}}$ is a continuous function that satisfies:

– $\kappa_\varphi(a)_z \neq \kappa_\varphi(a)_{z+1}$ only if $z \in \mathbb{D}(a)$;
– if $\forall z \in [d+1, s(d)], \kappa_\varphi(a)_z = k$, then $\exists b \in \Sigma, \varphi(b) = k$ and $a_{[d+1,s(d)]} \sqsubset_{d+1} b$
– $\kappa_\varphi(\sigma(a))_z = \kappa_\varphi(a)_z + 1$

When $s(d) > d + r - 2$, the second condition is equivalent to $\kappa_\varphi(a)_z = \varphi(a_{[d+1,d+r-1]})+d+1$ and shows that the phase signature is defined locally. Since we want our classification of defects to be σ-invariant, and considering the last condition, we define $\mathbb{D}^{\kappa_\varphi}_{i,j}(a) = \{d \in \mathbb{D}(a) : \kappa_\varphi(a)_d + d = i, \kappa_\varphi(a)_{d+1} + d + 1 = j\}$. These sets can be extended to defects in finite words except for the first $r - 2$ and last $r - 1$ cells.

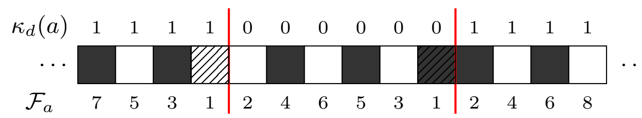

Fig. 3. Dislocations in the checkerboard subshift

3.4 Dynamics

We now consider the general case: assume that Σ can be decomposed into disjoint transitive SFTs $\Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ of respective periods $P_1 \ldots P_n$ (possibly 1). Moreover, we suppose that the Σ_i are F-invariant to give sense to the notion of dynamics of defects. $\kappa = (\kappa_d, \kappa_\varphi) : \mathcal{A}^\mathbb{Z} \to S^\mathbb{Z}$, where $S = \{(i,x) : i \in [1;n], x \in \mathbb{Z}/P_i\mathbb{Z}\}$ is a generalized *signature* if κ_d and κ_φ respect the conditions of sections 3.2 and 3.3, respectively. We classify the defects according to the signature of the surrounding phases $\mathbb{D} = \bigcup_{s_1,s_2 \in S} \mathbb{D}_{s_1,s_2}$ and there is some α such that we can extend this classification to finite words except for the first and last α cells.

To describe the dynamics of defects, we study the evolution from $\mathbb{D}(a)$ to $\mathbb{D}(F(a))$ for $a \in \mathcal{A}^\mathbb{Z}$. An *interpretation of the action of F on a configuration* $a \in \mathcal{A}^\mathbb{Z}$ is a function $\psi_a : \mathbb{D}(a) \to \mathcal{I}(\mathbb{D}(F(a)))$, where $\mathcal{I}(\mathbb{U})$ is the set of intervals of $\mathbb{U}$, that satisfies:

Locality $\forall d \in \mathbb{D}(a), \forall d' \in \psi_a(d), |d' - d| \leq r$;
Growth If $d_1 < d_2 \in \mathbb{D}(a)$, then $\psi_a(d_1) = \psi_a(d_2)$ or $\max(\psi_a(d_1)) < \min(\psi_a(d_2))$ (where $\max \emptyset = -\infty$ and $\min \emptyset = \infty$).
Surjectivity $\mathbb{D}(F(a)) = \bigcup_{d \in \mathbb{D}(a)} \psi_a(d)$.

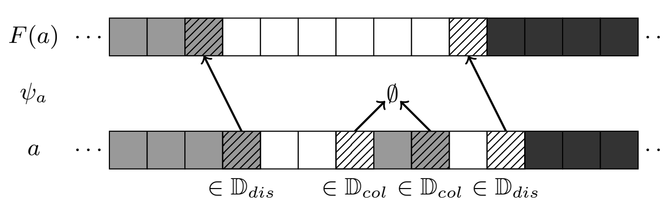

Fig. 4. An interpretation for the 3-state cyclic automaton

Equivalently, an interpretation is a decomposition of $\mathbb{D}(a)$ and $\mathbb{D}(F(a))$ into disjoint "increasing" intervals I_k and I'_k ($k \in \mathbb{Z}$) of size $\leq 2r+1$, satisfying locality (we have $\psi_a(I_k) = I'_k$). We distinguish different situations:

<table>
<tr><td>$|I'_k|$ / $|I_k|$</td><td>1</td><td>>1</td><td>0</td></tr>
<tr><td>1</td><td>displacement</td><td>explosion</td><td rowspan="2">(destructive) collision</td></tr>
<tr><td>>1</td><td>collision</td><td>collision</td></tr>
</table>

Thus, we decompose $\mathbb{D}(a)$ into $\mathbb{D}_{dis}(a) \sqcup \mathbb{D}_{col}(a) \sqcup \mathbb{D}_{expl}(a)$. If we have a set of interpretations $(\psi_a)_{a\in\mathcal{A}^{\mathbb{Z}}}$, we consider the iterated interpretation $\psi_a^2 : d \mapsto \bigcup_{d'\in\psi_a(d)} \psi_{F(a)}(d')$, which is an interpretation for F^2, and it extends to $n > 2$.

An interpretation ψ_a is *coalescent* if it contains only displacements ($|I_k| = |I'_k| = 1$) and decreasing collisions ($|I_k| > |I'_k|$). In this case, a defect $d \in \mathbb{D}_{s_1,s_2}(a)$ has *speed* $(p,q) \in \mathbb{Z}\times\mathbb{N}^*$ if $\forall k < q$, $\psi_a^k(d) \subseteq \mathbb{D}_{dis}(F^k(a))$ and $\psi_a^q(d) = \{d+p\} \subseteq \mathbb{D}_{s_1,s_2}(F^k(a))$. An interpretation ψ_a respects a *velocity function* $V : S^2 \to \mathbb{Z}\times\mathbb{N}^*$ if for any $s_1, s_2 \in S$ and any $d \in \mathbb{D}_{s_1,s_2}(a)$, writing $V(s_1,s_2) = \frac{p}{q}$, either $\psi_a^k(d) \subseteq \mathbb{D}_{col}(F^k(a))$ for some $k < q$, or d has speed $V(s_1,s_2)$. The *order* of a velocity function is the least common multiple of the q appearing in the image of V.

4 A step towards self-organization

Proposition 1. *Let F be a CA, $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ a decomposition into F-invariant SFTs, κ a signature, $a \in \mathcal{A}^{\mathbb{Z}}$ and ψ_a a coalescent interpretation. Then,*

1. *$d_{\mathbb{D}}(F(a)) \leq d_{\mathbb{D}}(a) - \frac{1}{2r+1} d_{\mathbb{D}_{col}}(a)$;*
2. *if ψ_a respects V of order q, $d_{\mathbb{D}_{s_1,s_2}}(F^q(a)) \leq d_{\mathbb{D}_{s_1,s_2}}(a) + \sum_{k=0}^{q} d_{\mathbb{D}_{col}}(F^k(a))$.*

Proof (of 1). By surjectivity and locality,

$$\forall n \in \mathbb{N}, \mathbb{D}(F(a)) \cap \mathbb{B}_n \subseteq \psi_a(\mathbb{D}(a) \cap \mathbb{B}_{n+r}).$$

Besides, $|\psi_a(\mathbb{D}_{dis}(a) \cap \mathbb{B}_{n+r})| = |\mathbb{D}_{dis}(a) \cap \mathbb{B}_{n+r}|$, and $|\psi_a(\mathbb{D}_{col}(a) \cap \mathbb{B}_{n+r})| \leq \frac{2r}{2r+1} |\psi_a^{-1}(\psi_a(\mathbb{D}_{col}(a) \cap \mathbb{B}_n))| \leq \frac{2r}{2r+1} |\mathbb{D}_{col}(a) \cap \mathbb{B}_{n+2r}|$.

Since the automaton is coalescent, there is no defect in explosion. Therefore, we have $\forall n \in \mathbb{N}, |\mathbb{D}(F(a)) \cap \mathbb{B}_{n+r}| \leq |\mathbb{D}_{dis}(a) \cap \mathbb{B}_{n+r}| + \frac{2r}{2r+1} |\mathbb{D}_{col}(a) \cap \mathbb{B}_{n+2r}|$, and we conclude by passing to the upper limit. □

Proof (of 2). We only prove the case q=1. Again,

$$\forall n \in \mathbb{N}, \mathbb{D}_{s_1,s_2}(F(a)) \cap \mathbb{B}_n \subseteq \psi_a(\mathbb{D}(a) \cap \mathbb{B}_{n+r}).$$

If $d \in \mathbb{D}_{dis}(a)$ and $\psi_a(d) \in \mathbb{D}_{s_1,s_2}(F(a))$, we have $d \in \mathbb{D}_{s_1,s_2}(a)$ since ψ_a respects V. Since $\mathbb{D}(a) = \mathbb{D}_{dis} \sqcup \mathbb{D}_{col}$, we conclude by passing to the upper limit. □

Theorem 1 (Main result). *Let F be a CA, $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$ a decomposition into σ-transitive F-invariant SFTs, κ a signature and $(\psi_a)_{a\in\mathcal{A}^{\mathbb{Z}}}$ a set of coalescent interpretations that respect a velocity function V.*

Then for all $\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$, there is a speed $v \in \mathbb{Q}$ such that

$$\forall s_1, s_2 \in S, \forall a \in \Lambda_\mu(F), \mathbb{D}_{s_1,s_2}(a) \neq \emptyset \Rightarrow V(s_1,s_2) = (p,q) \text{ with } \frac{p}{q} = v.$$

We only prove the result for velocity functions of order 1; the result can be easily extended by considering F^q. We note $V(s_1,s_2) = p$ instead of $(p,1)$. Let α be the order of Σ and κ a signature, we introduce the following sets of words:

$$\begin{aligned} I_{s_1,s_2} &= \{u \in \mathcal{A}^{2\alpha+1} : \alpha+1 \in \mathbb{D}_{s_1,s_2}(u)\} \\ J_{s_1,s_2,s_3,s_4}(n) &= \{u \in \mathcal{A}^{n+2\alpha+1} : \alpha+1 \in \mathbb{D}_{s_1,s_2}(u), n+\alpha+1 \in \mathbb{D}_{s_3,s_4}(u)\} \\ I &= \bigcup_{s_1,s_2\in S} I_{s_1,s_2} \end{aligned}$$

It is obvious that for $a \in \mathcal{A}^{\mathbb{Z}}$, for all $s_1, s_2 \in S, d_{\mathbb{I}_{s_1,s_2}(a)} = d_{\mathbb{D}_{s_1,s_2}(a)}$ (recall that $\mathbb{U}$ is the set of positions of U).

Lemma 1. *Under the previous assumptions, let $s_i \in S$ such that $V(s_1,s_2) > V(s_3,s_4)$. Then:*

$$\forall a \in \mathcal{A}^{\mathbb{Z}}, \forall n \in \mathbb{N}, d_{\mathbb{D}}(F^n(a)) \leq d_{\mathbb{D}}(a) - \frac{1}{4r+2} d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(n)}(a).$$

Proof. For $a \in \mathcal{A}^{\mathbb{Z}}$, we proceed by induction on n.

• *Initialization ($n = 1$):* let $x \in \mathbb{J}_{s_1,s_2,s_3,s_4}(1)$, that is $z = x+\alpha+1 \in \mathbb{D}_{s_1,s_2}(a)$ and $z+1 \in \mathbb{D}_{s_3,s_4}(a)$.

Assume that $z, z+1 \in \mathbb{D}_{dis}(a)$: then $\psi_a(z) = z + V(s_1,s_2) \geq \psi_a(z+1) = z+1+V(s_3,s_4)$, which is a contradiction with the growth property. Therefore, either z or $z+1$ is in collision, and $d_{\mathbb{D}_{col}}(a) \geq \frac{1}{2} d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(1)}(a)$. We conclude by Proposition 1 (1).

• *Heredity ($n > 1$):* we assume the lemma is true for all $k < n$, and we consider $x \in \mathbb{J}_{s_1,s_2,s_3,s_4}(n)$, that is $z = x+\alpha+1 \in \mathbb{D}_{s_1,s_2}(a)$ and $z+n \in \mathbb{D}_{s_3,s_4}(a)$.

Assume that $z, z+n \in \mathbb{D}_{dis}(a)$: then $\psi_a(z) = z + V(s_1,s_2)$ and $\psi_a(z+n) = z+n+V(s_3,s_4)$, and so $x + V(s_1,s_2) \in \mathbb{J}_{s_1,s_2,s_3,s_4}(k)(F(a))$ where $k = n - V(s_1,s_2) + V(s_3,s_4) < n$. We conclude that $z \in \mathbb{D}_{col}(a)$ or $z+n \in \mathbb{D}_{col}(a)$ or $z, z+n \in \mathbb{D}_{dis}(a)$ and $x + V(s_1,s_2) \in \mathbb{J}_{s_1,s_2,s_3,s_4}(k)(F(a))$.

Therefore, we have $d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(n)}(a) \leq 2d_{\mathbb{D}_{col}}(a) + d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(k)}(F(a))$.

If we apply the induction hypothesis,

$$\begin{aligned} d_{\mathbb{D}}(F^{k+1}(a)) &\leq d_{\mathbb{D}}(F(a)) - \frac{1}{4r+2} d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(k)}(F(a)) \\ &\leq d_{\mathbb{D}}(a) - \frac{1}{2} d_{\mathbb{D}_{col}}(a) - \frac{1}{4r+2} d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(k)}(F(a)) \\ &\leq d_{\mathbb{D}}(a) - \frac{1}{4r+2} d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(k)}(a) \end{aligned}$$

Since $d_{\mathbb{D}}(F^n(a)) \leq d_{\mathbb{D}}(F^{k+1}(a))$ (Proposition 1 (1)), we conclude. $\square$

Proof (of Theorem 1). By Birkhoff's theorem, we have for almost all $a \in \mathcal{A}^{\mathbb{Z}}$, $F^n\mu([I]) = d_{\mathbb{D}}(F^n(a))$. By prop. 1 (1), $(F^n\mu([I]))_{n\in\mathbb{N}}$ is decreasing and has a limit d_∞.

First step. Assume $\exists s_i \in S, V(s_1,s_2) > V(s_3,s_4)$ and $F^n\mu([I_{s_1,s_2}]) \cdot F^n\mu([I_{s_3,s_4}]) \not\to 0$; let $\varepsilon > 0$ such that $\forall n_0, \exists n \geq n_0, F^n\mu([I_{s_1,s_2}]) \cdot F^n\mu([I_{s_3,s_4}]) > \varepsilon$.

Consider n large enough so that $(F^n\mu([I])) - d_\infty < \frac{\varepsilon}{8r+4}$ and $F^n\mu([I_{s_1,s_2}]) \cdot F^n\mu([I_{s_3,s_4}]) > \varepsilon$. Since $F^n\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$, we have:

$$\frac{1}{N}\sum_{k=0}^{N} F^n\mu([I_{s_1,s_2}] \cap \sigma^k([I_{s_3,s_4}])) \underset{N\to\infty}{\longrightarrow} F^n\mu([I_{s_1,s_2}]) \cdot F^n\mu([I_{s_3,s_4}])$$

$$\frac{1}{N}\sum_{k=0}^{N} F^n\mu([J_{s_1,s_2,s_3,s_4}(k)]) \underset{N\to\infty}{\longrightarrow} F^n\mu([I_{s_1,s_2}]) \cdot F^n\mu([I_{s_3,s_4}])$$

We can choose N large enough so that $\frac{1}{N}\sum_{k=0}^{N} F^n\mu([J_{s_1,s_2,s_3,s_4}(k)]) > \frac{\varepsilon}{2}$, so we have $F^n\mu([J_{s_1,s_2,s_3,s_4}(k_0)]) > \frac{\varepsilon}{2}$ for some k_0.

By the preliminary lemma, we have:

$$\forall a \in \mathcal{A}^{\mathbb{Z}}, \qquad d_{\mathbb{D}}(F^{n+k_0}(a)) \leq d_{\mathbb{D}}(F^n(a)) - \frac{1}{4r+2} d_{\mathbb{J}_{s_1,s_2,s_3,s_4}(k_0)}(F^n(a))$$

$$F^{n+k_0}\mu([I]) \leq F^n\mu([I]) - \frac{\varepsilon}{8r+4}$$

Which is in contradiction with $F^n\mu([I]) - d_\infty < \frac{\varepsilon}{8r+4}$.

Second step. Assume $\exists s_i \in S$ such that $V(s_1,s_2) > V(s_3,s_4)$ and $F^n\mu([I_{s_1,s_2}]) \not\to 0, F^n\mu([I_{s_3,s_4}]) \not\to 0$. Since $F^n\mu([I_{s_1,s_2}]) \cdot F^n\mu([I_{s_3,s_4}]) \to 0$, 0 is an accumulation point of at least one of the sequences. Let $\varepsilon > 0$: w.l.o.g, we can choose n large enough so that $F^n\mu([I]) - d_\infty < \frac{\varepsilon}{4r+2}$ and $F^n\mu([I_{s_1,s_2}]) \leq \frac{\varepsilon}{2}$.

By applying iteratively proposition 1 (1) and (2), we have

$$\forall a \in \mathcal{A}^{\mathbb{Z}}, \forall k \in \mathbb{N}, \qquad d_{\mathbb{D}_{s_1,s_2}}(F^k(a)) - d_{\mathbb{D}_{s_1,s_2}}(a) \leq (2r+1) \cdot \big(d_{\mathbb{D}}(a) - d_{\mathbb{D}}(F^k(a))\big)$$

$$\forall k \in \mathbb{N}, \qquad F^k\mu([I_{s_1,s_2}]) - \mu([I_{s_1,s_2}]) \leq (2r+1) \cdot \big(\mu([I]) - F^k\mu([I])\big)$$

By applying this to the measure $F^n\mu$, we have:

$$\forall k \in \mathbb{N}, F^{n+k}\mu([I_{s_1,s_2}]) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

From which we deduce $F^n\mu([I_{s_1,s_2}]) \underset{n\to\infty}{\longrightarrow} 0$, a contradiction.

Summary : For all s_1, s_2, s_3, s_4 such that $V(s_1,s_2) \neq V(s_3,s_4)$, $F^n\mu([I_{s_1,s_2}]) \to 0$ or $F^n\mu([I_{s_3,s_4}]) \to 0$. □

5 Applications

5.1 *n*-state cyclic automaton

The n-state cyclic automaton is a particular captive cellular automaton defined on the alphabet $\mathcal{A} = \mathbb{Z}/n\mathbb{Z}$ by the local rule

$$f(a_{i-1}, a_i, a_{i+1}) = \begin{cases} a_i + 1 \text{ if } a_{i-1} = a_i + 1 \text{ or } a_{i+1} = a_i + 1 \\ a_i \qquad \text{otherwise} \end{cases}$$

This automaton was introduced by [Fis90]. In this paper, the author shows that for all Bernoulli measure μ, the set $[i]_0$ (for $i \in \mathcal{A}$) is a μ-attractor iff $n \geq 5$. Simulations starting from a random configuration suggest the following: for $n = 3$ or 4, monochromatic regions keep increasing in size; for $n \geq 5$, we observe the convergence to a fixed point where small regions are delimited by vertical lines. We are going to apply the main result to explain this observation.

We consider the decomposition $\Sigma = \bigsqcup_{i\in\mathcal{A}} \Sigma_i$ where $\Sigma_i = \{^\infty i^\infty\}$ of periods $P_i = 1$ (no dislocations). Here, $\kappa_d(a,z) = a_z$ and since $\kappa_\varphi = 1$, we write $\kappa(a,z) = i$ for $(i,1)$. Defects are exactly transitions between colors, and we define the velocity function as $V(i+1,i) = (1,1)$, $V(i,i+1) = (-1,1)$ and $V(i,j) = (0,1)$ for $i,j \in \mathcal{A}$ with $i+1 \neq j \neq i-1$.

For any $a \in \mathcal{A}^{\mathbb{Z}}$, we define $\mathbb{D}_k(a) = \bigcup_{V(i,j)=(k,1)} \mathbb{D}_{i,j}(a)$. We also define for any $a \in \mathcal{A}^{\mathbb{Z}}$ the function ψ_a by:

$$\forall d \in \mathbb{D}_k, \psi_a(d) = \begin{cases} \emptyset & \text{if } \exists d' \in \mathbb{D}_{k'}, sign(d-d') \neq sign(d+k-(d'+k')) \\ \{d+k\} & \text{otherwise} \end{cases}$$

This interpretation corresponds to the behaviour of defects as observed in simulations. It is straightforward to prove that it is well-defined (that is, it maps a defect to a interval of defects) and that it satisfies the properties of locality, growth and surjectivity. Since no image interval has size bigger than 1, it is coalescent and respects the velocity function V. By applying the main result, we show that for all $\mu \in \mathcal{M}_\sigma^{erg}(\mathcal{A}^{\mathbb{Z}})$, defects in only one direction remain in the μ-limit set, that is $\exists k \in \{-1,0,1\}, \forall a \in \Lambda_\mu(F), \mathbb{D}(a) = \mathbb{D}_k(a)$.

In particular, for any Bernoulli measure μ, if we consider the "mirror" application $\gamma((a_i)) = (a_{-i})$, we have $\mu(\gamma([u])) = \mu([u^{-1}]) = \mu([u])$, where $u_1 \dots u_n^{-1} = u_n \dots u_1$. But $d \in \mathbb{D}_1(a) \Leftrightarrow -d \in \mathbb{D}_{-1}(\gamma(a))$, and conversely; since this is true for any $F^k\mu$, one has $\mathbb{D}_1(a) = \mathbb{D}_{-1}(a) = \emptyset$ for all $a \in \Lambda_\mu(F)$. We deduce the following properties of Λ_μ for each n-cyclic cellular automaton:

- If $n = 3$, there is no defect of speed 0. Therefore, one has $\mathbb{D}(a) = \emptyset$ for all $a \in \Lambda_\mu(F)$, which means that $\Lambda_\mu(F) = \Sigma$ is a set of monochromatic configurations.
- If $n = 4$, the result of [Fis90] shows that $[i]_0$ cannot be a μ-attractor for all i. Thus one has $\mathbb{D}_0(a) = \emptyset$ for all $a \in \Lambda_\mu(F)$, and $\Lambda_\mu(F) = \Sigma$ is a set of monochromatic configurations.
- If $n \geq 5$, the result of [Fis90] shows that $[i]_0$ is a μ-attractor for all i. Thus for some $a \in \Lambda_\mu(F)$ one has $\mathbb{D}(a) = \mathbb{D}_0(a) \neq \emptyset$. This means that they contain homogeneous regions separated by vertical lines.

5.2 Automaton #184

On the #184 "traffic" cellular automaton, we consider the defects according to $\Sigma = \{^\infty(01)^\infty, {}^\infty(10)^\infty\}$ (checkerboard pattern) with $P = 2$ (no interfaces). Since $\kappa_d = 1$, we write $\kappa(a,z) = i$ for $(1,i)$. If we define the phases $\varphi(^\infty(01)^\infty) = 0$ and $\varphi(^\infty(10)^\infty) = 1$, we can see that κ is unambiguous and that $\kappa(a,z) = 0$ if

$a_z = z \mod 2$, 1 otherwise. We define the velocity function as $V(0,0) = (1,1)$ and $V(1,1) = (-1,1)$ (this corresponds to $\{\square\square\}$ and $\{\blacksquare\blacksquare\}$, respectively).

For any $a \in \mathcal{A}^{\mathbb{Z}}$, we define ψ_a by

$$\forall d \in \mathbb{D}_{0,0}, \psi_a(d) = \begin{cases} \emptyset & \text{if } d+2 \in \mathbb{D}_{1,1} \\ \{d+1\} & \text{otherwise} \end{cases}$$

And symetrically for $d \in \mathbb{D}_{1,1}(a)$. Similarly, we can check that this interpretation is well-defined, respects the properties of locality, surjectivity and growth, is coalescent and respects the velocity function V. By applying the previous theorem, we have for all $\mu \in \mathcal{M}_{\sigma}^{erg}(\mathcal{A}^{\mathbb{Z}})$ either $\mathbb{D}_{0,0}(a) = \emptyset$ (checkerboard and monochromatic black patterns) or $\mathbb{D}_{1,1}(a) = \emptyset$ (checkerboard and monochromatic white patterns).

In particular, for the uniform Bernoulli measure μ, we consider the application $\gamma'((a_i)) = (1 - a_{-i})$, and we can see that $\mu(\gamma'([u])) = \mu([\overline{u^{-1}}]) = \mu([u])$, where $\overline{u_1 \dots u_n} = (1-u_1)\dots(1-u_n)$. But $d \in \mathbb{D}_{0,0}(a) \Leftrightarrow -d \in \mathbb{D}_{1,1}(\gamma'(a))$, and conversely; therefore, for all $a \in \Lambda_{\mu}(F)$, $\mathbb{D}_{0,0}(a) = \mathbb{D}_{1,1}(a) = \emptyset$. We deduce that $\Lambda_{\mu}(F)$ is the checkerboard subshift and by σ-invariance, $(F^n\mu)$ admits a single accumulation point $\frac{1}{2}\delta_{{}^{\infty}01^{\infty}} + \frac{1}{2}\delta_{{}^{\infty}10^{\infty}}$.

5.3 Captive one sided cellular automata

Now consider a captive cellular automaton $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ of neighborhood $[0;1]$, which means that the local rule $f : \mathcal{A}^{[0;1]} \to \mathcal{A}$ verifies $f(a_0a_1) \in \{a_0, a_1\}$. Captive cellular automata were introduced in [The04] and have some interesting algebraic propeties.

We consider the decomposition $\Sigma = \bigsqcup_{i \in \mathcal{A}} \Sigma_i$ where $\Sigma_i = \{{}^{\infty}i^{\infty}\}$ of periods $P_i = 1$ (no dislocations). We define the velocity function as $V(i,j) = (-1,1)$ if $f(ij) = j$ and $V(i,j) = (0,1)$ if $f(ij) = i$, and we define $\mathbb{D}_{-1}$ and $\mathbb{D}_0$ as in 5.1. For all $a \in \mathcal{A}^{\mathbb{Z}}$, we define:

$$\forall d \in \mathbb{D}_{-1}(a), \psi_a(d) = \begin{cases} \emptyset & \text{if } d-1 \in \mathbb{D}_0 \\ \{d-1\} & \text{otherwise} \end{cases}$$

and symetrically if $d \in \mathbb{D}_0$. As in the two previous examples, we can check that this is well-defined and respects the properties of locality, growth, surjectivity, coalescence and the velocity function.

Thus, for any σ-ergodic measure μ, $\Lambda_{\mu}(F)$ contains defects in one direction only. If moreover, for all $a, b \in \mathcal{A}$, the local rule verifies $f(ab) = f(ba)$ and μ verifies $\mu([ab]) = \mu([ba])$ (e.g. Bernoulli measures), we have $\Lambda_{\mu}(F) = \Sigma$.

6 Conclusion

In this article we have presented a formalism to link the notion of defect with respect to a subshift Σ introduced by M. Pivato and the emergence of homogeneous regions separated by defects when we iterate a random configuration.

Under some assumptions on the collisions of defects, we proved the only defects that possibly remain in the μ-limit set all have the same direction. This explains the behaviour observed in simulations for large classes of cellular automata.

Acknowledgments

The authors are grateful to Guillaume Theyssier for pointing out the problematic around the captive cellular automata class. We also thank the project ANR EMC: ANR-09-BLAN-0164 for financial support.

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